

Exponential Metric Fields

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Abstract The Laser Interferometer Space Antenna (LISA) mission will use advanced technologies to achieve its science goals: the direct detection of gravitational waves, the observation of signals from compact (small and dense) stars as they spiral into black holes, the study of the role of massive black holes in galaxy evolution, the search for gravitational wave emission from the early Universe. The gravitational red-shift, the advance of the perihelion of Mercury, deflection of light and the time delay of radar signals are the classical tests in the first order of General Relativity (GR). However, LISA can possibly test Einstein's theories in the second order and perhaps, it will show some particular feature of non-linearity of gravitational interaction. In the present work we are seeking a method to construct theoretical templates that limit in the first order the tensorial structure of some metric fields, thus the non-linear terms are given by exponential functions of gravitational strength. The Newtonian limit obtained here, in the first order, is equivalent to GR.

Keywords Linearized gravity, Newtonian limit, Gravitational waves

1 Introduction

The extreme difficulties which arise if one tries to draw physically important conclusions from the basic assumptions of Einstein's theory are mainly due to the non-linearity of the field equations. Moreover, the fact that the spacetime topology is not given a priori, and the impossibility to integrate tensors over finite regions cause difficulties unknown in other branches of mathematical physics. Actually in this respect they are not

so different from others fields, for example the electromagnetic field, the scalar field, etc., by themselves obey linear equations in a given spacetime, they form a non-linear system when their mutual interactions are taken into account. The distinctive feature of the gravitational field is that it is self-interacting (as the Yang-Mills field): it is non-linear even in the absence of other fields. This is because it defines the spacetime over which it propagates (Hawking & Ellis 1973).

Linearized gravity is any approximation to General Relativity obtained from $g_{\mu\nu} = g_{\mu\nu}^{(0)} + h_{\mu\nu}$ (where $g_{\mu\nu}^{(0)}$ is any curved background spacetime) in Einstein's equation and retaining only the terms linear in $h_{\mu\nu}$ (Wald 1984). The weakness of the gravitational field means in the context of general relativity that the spacetime is nearly flat. Small gravitational perturbations in Minkowski space can be treated in the simplest linearized version of General Relativity,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad (1)$$

as describing a theory of a symmetric tensor field $h_{\mu\nu}$ propagating on at background spacetime. This theory is Lorentz invariant in the sense of Special Relativity. If one wants to obtain a solution of the non-linear equations, it is necessary to employ an iterative method on approximate linear equations whose solutions are shown to converge in a certain neighbourhood of initial surface. It should be possible to avoid some of these difficulties of non-linearity by working in some spacetimes that shall be described in this paper. The proposal is that some metric fields can be separated into the parts carrying the dynamical information and those parts characterizing the coordinates system. In this proposal, terms of the coordinates system will have tensorial structure limited only in the first order. The tensors that will describe $g_{\mu\nu}$ have linear behavior. Naturally, there is a price that must be paid for the linear

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tensors. The dynamic terms that carry the gravitational strength have exponential structure. The principal idea came from the basic principle that one should interpret (1) as separation between pure mathematical and physical terms in metric field tensor. In spite of the fact that $\eta_{\mu\nu}$ plays a key role as empty flat and background spacetime of the Standard Model in the description of fundamental interactions, this background tensor metric $\eta_{\mu\nu}$ is an object wholly mathematical and entirely geometrical, while $h_{\mu\nu}$ contains the physical information. The strength of gravity is tied in the components of $h_{\mu\nu}$. The proposal of this paper is a working hypothesis to untie the strength of gravity from geometrical tensors. This proposal is valid for a family of metric field tensors $g_{\mu\nu}$, and some basic examples such Newtonian limit and gravitational plane waves of low amplitudes are described.

This paper is outlined as follows: in Sec. II, we present the basic mathematic concepts of the (quasi-)idempotent tensors that compose the structure of metric fields approached in this work. In Sec. III, we propose how to link the strength of gravity with the tensors from Sec. II, then is defined a family of exponential metrics. In Sec. IV, we present some examples of these exponential metrics, such as: Yilmaz metric, circularly polarized wave and rotating bodies. In Sec. V, we present exponential metrics ('adjoint metric field') that are non-physical, but which help us to compute Christoffel symbols, and consequently the curvature tensors, Ricci tensors and determinant of metric field. In Sec. VI, we verify the Newtonian limit and also we obtain gravitational waves. In Sec. VII, we present a general conclusion.

We assume spacetime $(\mathcal{M}, \mathbf{g})$ to be a $C^\infty 4$ -dimensional, globally hyperbolic, pseudo-Riemannian manifold \mathcal{M} with Lorentzian metric tensor \mathbf{g} (whose components are $g_{\mu\nu}$) associated with the line element

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu,$$

assumed to have signature $(+ - - -)$ (Landau & Lifschitz 1984). Lower case Greek indices refer to coordinates on \mathcal{M} and take the values 0, 1, 2, 3. The relation between the metric field $g_{\mu\nu}$ and the material contents of spacetime is expressed by Einstein's field equation,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu}, \quad (2)$$

$T_{\mu\nu}$ being the stress-energy-momentum tensor, $R_{\mu\nu}$ the contract curvature tensor (Ricci tensor) and R its trace. In an empty region of spacetime we have $R_{\mu\nu} = 0$, such a region is called vacuum field.

2 Pure Mathematical terms of spacetime geometry

The Minkowski flat spacetime $(\mathbb{R}^4, \boldsymbol{\eta})$, where the components of $\boldsymbol{\eta}$ are $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$, is the simplest empty spacetime and it is in fact the spacetime of Special Relativity. One can obtain spaces locally identical to $(\mathbb{R}^4, \boldsymbol{\eta})$ but with different (large scale) topology properties by identifying points in \mathbb{R}^4 which are equivalent under a discrete isometry without a fixed point. Any local Lorentz frames is merely the statement that any curved space has the Minkowski flat space 'tangent' to it at any point. The Minkowski spacetime is the universal covering space for all such derived spaces (Hawking & Ellis 1973). In this sense it should be reasonable to one choose $(\mathbb{R}^4, \boldsymbol{\eta})$ as background spacetime and important compound piece of some spacetimes. In fact, the most straightforward approach to linear gravitation is realized in Minkowski spacetime. The conformal structure of Minkowski space is what one would regard as the 'normal' behavior of a spacetime at infinity.

The metric tensor $\boldsymbol{\eta}$ from background Minkowski spacetime in any coordinates is an object wholly mathematical and entirely geometrical. No strength of gravity is linked up with the mathematical structure of $\boldsymbol{\eta}$. Then, we choose this metric tensor as a principal descriptive piece of some spacetimes built below, and it does rise and lower the indices in the same way as in Special Relativity with $\eta_{\mu\nu}\eta^{\nu\alpha} = \delta_\mu^\alpha$. The aim of this paper is to obtain some spacetimes $(\mathcal{M}, \mathbf{g})$ that their non-linearity are less hard, softer at least in a tensorial descriptive way. Therefore, it is defined only a symmetric tensor $\boldsymbol{\Upsilon}$, which like $\boldsymbol{\eta}$ is an object wholly mathematical and entirely geometrical. This (metric) tensor $\boldsymbol{\Upsilon}$ that will be a piece of metric \mathbf{g} can have non-static terms from spacetime in their components, however in this approach, this tensor purely will not have gravitational strength. The components $\Upsilon_{\mu\nu}$ of tensor $\boldsymbol{\Upsilon}$ are raised and lowered by $\eta^{\mu\nu}$ and $\eta_{\mu\nu}$,

$$\begin{aligned} \eta^{\mu\nu}\Upsilon_{\nu\alpha} &= \Upsilon^\mu_\alpha, \\ \Upsilon_{\mu\nu}\eta^{\nu\alpha} &= \Upsilon_\mu^\alpha, \\ \eta^{\mu\nu}\Upsilon_{\nu\alpha}\eta^{\alpha\beta} &= \Upsilon^{\mu\beta}. \end{aligned} \quad (3)$$

In this context it is adopted the point of view, that $\boldsymbol{\Upsilon}$ is a tensor on a background Minkowski spacetime, similar to deviation $h_{\mu\nu}$ from linearized version of general relativity (1). But instead one has the infinitesimal condition for $|h_{\mu\nu}| \ll 1$, it is accepted that the magnitude of $\Upsilon_{\mu\nu}$ can be equal to the magnitude of empty flat spacetime ($|\Upsilon_{\mu\nu}| \approx |\eta_{\mu\nu}|$). Moreover, it might be defined as an important mathematical relationship among

$\Upsilon_{\mu\nu}$ themselves,

$$\Upsilon_{\mu\nu}\Upsilon^{\nu\rho} = -2\Upsilon_{\mu}^{\rho}. \quad (4)$$

The above equation is an important argument to shape some spacetimes that are described below. This equation will improve linearity in the tensorial sense. The contracting indices of tensor Υ by themselves show that $\Upsilon_{\mu\nu}$ are (quasi-)idempotent elements ($\Upsilon \cdot \Upsilon \propto \Upsilon$; otherwise a factor -2 in the operation), and the equation (4) improves at least to a linear tensorial template of some tensor metric fields. It is possible to rise or lower indices of $\Upsilon_{\mu\nu}$ operating themselves,

$$\Upsilon^{\rho\mu}\Upsilon_{\mu\nu}\Upsilon^{\nu\sigma} = 4\Upsilon^{\rho\sigma}. \quad (5)$$

One can also verify the expression $\Upsilon_{\mu}^{\nu}\Upsilon_{\nu\rho} = -2\Upsilon_{\mu\rho}$.

The trace of Υ is obtained when $\rho = \mu$ in the expression (4),

$$\Upsilon_{\mu\nu}\Upsilon^{\nu\mu} = -2\Upsilon_{\mu}^{\mu}, \quad (6)$$

and realize derivative calculation of trace Υ_{μ}^{μ} from (6) since $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$,

$$-2\partial_{\alpha}\Upsilon_{\mu}^{\mu} = 2\Upsilon_{\mu\nu}\partial_{\alpha}\Upsilon^{\mu\nu}, \quad (7)$$

thus,

$$\Upsilon_{\mu\nu}\partial_{\alpha}\Upsilon^{\mu\nu} = \Upsilon^{\mu\nu}\partial_{\alpha}\Upsilon_{\mu\nu} = -\partial_{\alpha}\Upsilon_{\mu}^{\mu}. \quad (8)$$

Only both these tensors, η from background Minkowski spacetime and Υ , will become the mathematical and geometrical basis to physical spacetimes described as follows.

3 Strength of gravity terms tied in spacetime geometry

A dimensionless parameter Φ characterizing the strength of gravity at a spacetime point \wp with coordinates $(t, \mathbf{x}) = (x^{\alpha})$ due to a gravitating source is the ratio of the potential energy, $m\varphi_N$ (due to this source), to the inertial mass-energy mc^2 of a test body at \wp , i.e.,

$$\Phi(x^{\alpha}) = \frac{\varphi_N(x^{\alpha})}{c^2}. \quad (9)$$

Here $\varphi_N(x^{\alpha})$ is the gravitational potential. For a point source with mass M in Newtonian gravity,

$$\Phi(x^{\alpha}) = -\frac{GM}{c^2 r}, \quad (10)$$

where r is the distance to the source. So, for a nearly Newtonian system, we can use Newtonian potential for φ_N .

To construct spacetimes with the basis η from background Minkowski spacetime and Υ defined in previous section, it is proposed to tie the strength of gravity Φ to these tensors. While the tensor Υ can be a function of the coordinates $(t, \mathbf{x}) = (x^{\alpha})$, the strength of gravity Φ will be a function of the coordinates and also of the Newtonian gravitational constant G and of a parameter of mass M . Thus, I do propose:

$$\mathbf{g}(\Phi, x) = e^{\Phi}\eta + \sinh(\Phi)\Upsilon,$$

with components

$$g_{\mu\nu} = e^{\Phi}\eta_{\mu\nu} + \sinh(\Phi)\Upsilon_{\mu\nu}. \quad (11)$$

The inverse tensor is just

$$g^{\mu\nu} = e^{-\Phi}\eta^{\mu\nu} - \sinh(\Phi)\Upsilon^{\mu\nu}. \quad (12)$$

If the tensor Υ is diagonal, then the inverse tensor of \mathbf{g} is

$$\mathbf{g}^{-1}(\Phi, x) = \mathbf{g}(-\Phi, x) = e^{-\Phi}\eta - \sinh(\Phi)\Upsilon. \quad (13)$$

From the above definitions it follows that $g_{\mu\nu}g^{\nu\alpha} = \delta_{\mu}^{\alpha}$, in fact

$$g_{\mu\nu}g^{\nu\alpha} = (e^{\Phi}\eta_{\mu\nu} + \sinh(\Phi)\Upsilon_{\mu\nu})(e^{-\Phi}\eta^{\nu\alpha} - \sinh(\Phi)\Upsilon^{\nu\alpha})$$

where the rightside is

$$\delta_{\mu}^{\alpha} + \sinh(\Phi)(e^{-\Phi}\Upsilon_{\mu\nu}\eta^{\nu\alpha} - e^{\Phi}\eta_{\mu\nu}\Upsilon^{\nu\alpha} - \sinh(\Phi)\Upsilon_{\mu\nu}\Upsilon^{\nu\alpha})$$

that is

$$\delta_{\mu}^{\alpha} + \sinh(\Phi)[-(e^{\Phi} - e^{-\Phi})\Upsilon_{\mu}^{\alpha} - \sinh(\Phi)\Upsilon_{\mu\nu}\Upsilon^{\nu\alpha}] = \delta_{\mu}^{\alpha} - \sinh^2(\Phi)(2\Upsilon_{\mu}^{\alpha} + \Upsilon_{\mu\nu}\Upsilon^{\nu\alpha}) \quad (14)$$

Using the expression (4) in the second term in the parenthesis ($\Upsilon_{\mu\nu}\Upsilon^{\nu\alpha} = -2\Upsilon_{\mu}^{\alpha}$) it results in,

$$g_{\mu\nu}g^{\nu\alpha} = \delta_{\mu}^{\alpha} - \sinh^2(\Phi)[2\Upsilon_{\mu}^{\alpha} + (-2)\Upsilon_{\mu}^{\alpha}], \quad (15)$$

finally one did prove this identity $g_{\mu\nu}g^{\nu\alpha} = \delta_{\mu}^{\alpha}$. Because of equation (4), there are no quadratic values naturally in Υ and consequently the non-linearity of metric field will be less complicated.

If one considers some spacetime $(\mathcal{M}, \mathbf{g})$ such as the same of \mathbf{g} from definition (11), one can observe that the first term $\gamma_{\mu\nu} = e^{\Phi}\eta_{\mu\nu}$ is a background spacetime conformally flat (where the Weyl tensor vanishes). In the previous section one has accepted that $|\Upsilon_{\mu\nu}| \approx |\eta_{\mu\nu}|$, but now because of $\sinh(\Phi)$ (that can be small) multiplying this tensor Υ , it can be understood as (small) disturbance $\mathcal{S}_{\mu\nu} = \sinh(\Phi)\Upsilon_{\mu\nu}$, such as,

$$g_{\mu\nu} = \gamma_{\mu\nu} + \mathcal{S}_{\mu\nu}, \quad (16)$$

if one is dealing with the Einstein's vacuum equation, the (small) disturbance that represents the gravitational wave can be separated from the conformally flat background spacetime $\gamma_{\mu\nu}$ (Birrell & Davies 1982).

4 Some Examples

4.1 Yilmaz Metric

As a first application of the metric field defined in previous sections, let us take $\Upsilon_{\mu\nu}$ to be:

$$\Upsilon_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \quad \Upsilon_{\mu}{}^{\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$$

and

$$\Upsilon^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \quad (17)$$

that $\Upsilon_{\mu\nu}\Upsilon^{\nu\alpha} = -2\Upsilon_{\mu}{}^{\alpha}$, with trace given $\Upsilon_{\mu\nu}\Upsilon^{\nu\mu} = -2\Upsilon_{\mu}{}^{\mu} = -2\mathbf{Tr}\Upsilon$, that $\mathbf{Tr}(\Upsilon) = -6$. Now we can display the tensor metric (11),

$$g_{\mu\nu} = e^{\Phi} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} + \sinh(\Phi) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

$$g_{\mu\nu} = \begin{pmatrix} e^{\Phi} & 0 & 0 & 0 \\ 0 & -e^{-\Phi} & 0 & 0 \\ 0 & 0 & -e^{-\Phi} & 0 \\ 0 & 0 & 0 & -e^{-\Phi} \end{pmatrix}, \quad (18)$$

since $-e^{\Phi} + 2\sinh(\Phi) = -e^{-\Phi}$. Then the line element is,

$$ds^2 = e^{\Phi} c^2 dt^2 - e^{-\Phi} (dx^2 + dy^2 + dz^2). \quad (19)$$

This metric field (18,19) has been proposed by Yilmaz (Yilmaz 1958, 1992, 1976, 1982, 1973, 1977, 1997). In the case of a mass singularity, $\Phi = -\frac{2GM}{c^2 r} \ll 1$ we have the far-field metric,

$$ds^2 = (1 - \frac{2GM}{c^2 r}) c^2 dt^2 - (1 + \frac{2GM}{c^2 r}) (dx^2 + dy^2 + dz^2). \quad (20)$$

This is to be contrasted with the Schwarzschild (in General Relativity, GR) line element in isotropic coordinates (Landau & Lifschitz 1984),

$$ds^2 = \left(\frac{1 + \Phi/4}{1 - \Phi/4} \right)^2 c^2 dt^2 - \left(1 - \frac{\Phi}{4} \right)^4 (dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2), \quad (21)$$

if we compare expansions with the line element of Yilmaz theory (19),
Yilmaz:

$$g_{00} = 1 + \Phi + \frac{\Phi^2}{2} + \frac{\Phi^3}{6} + \dots \quad g_{11} = -1 + \Phi - \frac{\Phi^2}{2} + \dots$$

GR:

$$g_{00} = 1 + \Phi + \frac{\Phi^2}{2} + \frac{3\Phi^3}{16} + \dots \quad g_{11} = -1 + \Phi - \frac{3\Phi^2}{8} - \dots,$$

g_{11} coefficients differing only in the second order terms, while g_{00} differing in the third order. Both, Yilmaz and GR, give observational indistinguishable predictions for red-shift, light bending and perihelion advance, but the Yilmaz metric does not admit black holes. Citing this property and assuming that Yilmaz theory is correct, Clapp (Clapp 1973) has suggest that a significant component of quasar red-shift may be gravitational. Robertson (Robertson 1999a,b) has suggested that some neutron stars and black hole candidates may be like 'Yilmaz stars'. Robertson argues that neutron star with mass $\sim 10M_{\odot}$ is found for Yilmaz metric while that an object of nuclear density greater than $\sim 2.8M_{\odot}$ should be a black hole in Schwarzschild metric. Ibison has tested Yilmaz theory by working out the corresponding Friedmann equations generated by assuming the Friedmann-Robertson-Walker cosmological metrics (Ibison 2006). There are a series of claims and counter-claims involving Fackerell (Fackerell 1996; Yilmaz 1994; Alley & Yilmaz 2000), and also Misner and Wyss (Wyss & Misner 1999; Misner 1999; Alley & Yilmaz 1999) about Yilmaz theory. At the present time both Yilmaz and Schwarzschild solutions give results in agreement with observation (Rosen 1974). However, it may be possible in the future, with LISA mission (NASA 2010; Baker et al. 2007), to distinguish between Yilmaz and Schwarzschild.

4.2 Circularly Polarized Wave

Gravitational waves are one of the most important predictions of General Relativity. Now we can try a solution of gravitational waves in z direction,

$$\Upsilon_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 + \cos \zeta & \sin \zeta & 0 \\ 0 & \sin \zeta & 1 - \cos \zeta & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix},$$

$$\Upsilon_{\mu}{}^{\nu} = \Upsilon_{\mu\alpha} \eta^{\alpha\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 - \cos \zeta & -\sin \zeta & 0 \\ 0 & -\sin \zeta & -1 + \cos \zeta & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} \quad (22)$$

and

$$\Upsilon^{\mu\nu} = \Upsilon_{\alpha\beta} \eta^{\alpha\mu} \eta^{\beta\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 + \cos \zeta & \sin \zeta & 0 \\ 0 & \sin \zeta & 1 - \cos \zeta & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \quad (23)$$

so $\Upsilon_{\mu\nu}\Upsilon^{\nu\alpha} = -2\Upsilon_{\mu}{}^{\alpha}$ is verified for the tensor above and $\mathbf{Tr}(\Upsilon) = -4$. Then, if one chooses $\zeta = \omega t - kz$, the metric field is a solution for a gravitational plane

wave $g_{\mu\nu} = e^\Phi \eta_{\mu\nu} + \sinh(\Phi) \Upsilon_{\mu\nu}$,

$$g_{\mu\nu} = \begin{pmatrix} e^\Phi & 0 & 0 & 0 \\ 0 & -\cosh(\Phi) & 0 & 0 \\ 0 & 0 & -\cosh(\Phi) & 0 \\ 0 & 0 & 0 & -e^\Phi \end{pmatrix} + \sinh(\Phi) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \cos \zeta & \sin \zeta & 0 \\ 0 & \sin \zeta & -\cos \zeta & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}. \quad (24)$$

Where the first term can be the background spacetime (asymptotically flat) and the second term is the disturbance in this background or in other words the circularly polarized radiation $h_{\mu\nu}^{TT}$ with amplitude $\sinh(\Phi)$. The gravitational wave polarization is important from astrophysical and cosmological viewpoints. A binary system of two stars in circular orbit one around the other is expected to emit circularly polarized waves in the direction perpendicular to the plane of the orbit (Schutz 1990). Moreover, the Big Bang left behind an echo in the electromagnetic spectrum, the cosmic microwave background, but the Big Bang most likely also left cosmological gravitational waves that will be possible to observe with the help of LISA (NASA 2010; Baker et al. 2007). Since cosmological gravitational waves propagate without significant interaction after they are produced, once detected they should provide a powerful tool for studying the early Universe at the time of gravitational wave generation (Buonanno 2004). Various mechanisms for cosmological gravitational wave generation have been proposed, and many of these state that the cosmological gravitational wave are circularly polarized. T. Kahnashvili et al (Kahnashvili et al. 2005) argued that helical turbulence produced during a first-order phase transition generated circularly polarized cosmological gravitational waves. Other physicists have said that the parity violation due to the gravitational Chern-Simons term in superstring theory can produce the primordial gravitational waves with circular polarization (Lue et al. 1999; Choi et al. 2000; Alexander et al. 2006; Satoh et al. 2007; Saito et al. 2007).

If we assume long distance from source, we can obtain plane wave solution with $\Phi \ll 1$ so that,

$$g_{\mu\nu} = (1 + \Phi) \eta_{\mu\nu} + \Phi \Upsilon_{\mu\nu} = \begin{pmatrix} 1 + \Phi & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 + \Phi \end{pmatrix} + \Phi \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \cos \zeta & \sin \zeta & 0 \\ 0 & \sin \zeta & -\cos \zeta & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

the second term is the circularly polarized radiation $h_{\mu\nu}^{TT}$. Because of the far distance from source, we have

the presence of g_{00} and g_{33} as static terms, weakly perturbed in these coordinates.

4.3 Rotating Bodies

The Kerr metric is important astrophysically since it is a good approximation to the metric of a rotating star at large distances. It is possible to obtain a kind Kerr metric from $g_{\mu\nu} = e^\Phi \eta_{\mu\nu} + \sinh(\Phi) \Upsilon_{\mu\nu}$, that in coordinates (t, x, y, z) the tensor $\Upsilon_{\mu\nu}$ is:

$$\begin{pmatrix} \cosh^2 \Lambda & -\sinh \Lambda \cosh \Lambda \sin \phi & \sinh \Lambda \cosh \Lambda \cos \phi & 0 \\ -\sinh \Lambda \cosh \Lambda \sin \phi & \sinh^2 \Lambda \sin^2 \phi & -\sinh^2 \Lambda \cos \phi \sin \phi & 0 \\ \sinh \Lambda \cosh \Lambda \cos \phi & -\sinh^2 \Lambda \cos \phi \sin \phi & \sinh^2 \Lambda \cos^2 \phi & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (25)$$

satisfying $\Upsilon_{\mu\nu} \Upsilon^{\nu\rho} = -2\Upsilon_\mu{}^\rho$ with $\text{Tr}(\Upsilon) = -2$. In this example we choose $\Upsilon_{33} = 0$, but if the choice was $\Upsilon_{33} = 2$, the above tensor still satisfy the algebra (4). One can change coordinates (t, x, y, z) to the Boyer-Lindquist coordinates (t, r, θ, ϕ) , with spatial part as flat space in ellipsoidal coordinates,

$$\begin{aligned} t &= t, \\ x &= \sqrt{r^2 + a^2} \sin \theta \cos \phi, \\ y &= \sqrt{r^2 + a^2} \sin \theta \sin \phi, \\ z &= r \cos \theta, \end{aligned} \quad (26)$$

The Minkowski tensor metric related to them is:

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{r^2 + a^2 \cos^2 \theta}{r^2 + a^2} & 0 & 0 \\ 0 & 0 & -r^2 - a^2 \cos^2 \theta & 0 \\ 0 & 0 & 0 & -(r^2 + a^2) \sin^2 \theta \end{pmatrix}, \quad (27)$$

we are assuming that the angle ϕ from tensor $\Upsilon_{\mu\nu}$ of (25) can be the same angle from transformations (26). So, this coordinate transformations will become tensor (25) in:

$$\Upsilon_{\mu\nu} = (-2) \begin{pmatrix} \cosh^2 \Lambda & 0 & 0 & \sinh \Lambda \cosh \Lambda R \sin \theta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sinh \Lambda \cosh \Lambda R \sin \theta & 0 & 0 & \sinh^2 \Lambda R^2 \sin^2 \theta \end{pmatrix} \quad (28)$$

where $R = \sqrt{r^2 + a^2}$. At the appendices, it is verified with details that the above tensor obeys the algebra (4) in the background Minkowski spacetime (27). Now we can choose a particular solution for this tensor choosing the geometric terms $\sinh \Lambda$ and $\cosh \Lambda$:

$$\sinh \Lambda = -\frac{a \sin \theta}{\rho} \quad \text{and} \quad \cosh \Lambda = \frac{\sqrt{r^2 + a^2}}{\rho}, \quad (29)$$

with

$$\rho^2 = r^2 + a^2 \cos^2 \theta \quad (30)$$

satisfying $\cosh^2 \Lambda - \sinh^2 \Lambda = 1$.

The physical terms that contain the strength of gravity e^Φ and $\sinh \Phi$ can be:¹

$$\Phi = \frac{Mr}{(r^2 + a^2)} \ll 1, \quad (31)$$

so that:

$$e^\Phi \approx 1 + \Phi = 1 + \frac{Mr}{(r^2 + a^2)}$$

and

$$\sinh(\Phi) \approx \Phi = \frac{Mr}{(r^2 + a^2)}.$$

We can compute each term of metric field $g_{\mu\nu}$ (for more details see appendices),

$$\begin{aligned} g_{00} &= \frac{\Delta - a^2 \sin^2 \theta}{\rho^2} + \frac{Mr}{(r^2 + a^2)}, \\ g_{03} &= g_{30} = \frac{2Mr a \sin^2 \theta}{\rho^2}, \\ g_{11} &= -\frac{\rho^2}{\Delta} + \frac{Mr \rho^2}{\Delta(r^2 + a^2)}, \\ g_{22} &= -\rho^2 - \frac{Mr \rho^2}{r^2 + a^2}, \\ g_{33} &= \frac{-\sin^2 \theta [(r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta]}{\rho^2} - Mr \sin^2 \theta, \end{aligned}$$

where we use the definition ,

$$\Delta = r^2 - 2Mr + a^2. \quad (32)$$

The metric tensor $g_{\mu\nu}$ in the matrix form is:

$$\begin{pmatrix} \frac{\Delta - a^2 \sin^2 \theta}{\rho^2} & 0 & 0 & \frac{2Mr a \sin^2 \theta}{\rho^2} \\ 0 & -\frac{\rho^2}{\Delta} & 0 & 0 \\ 0 & 0 & -\rho^2 & 0 \\ \frac{2Mr a \sin^2 \theta}{\rho^2} & 0 & 0 & \frac{-\sin^2 \theta [(r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta]}{\rho^2} \end{pmatrix} + \begin{pmatrix} \frac{Mr}{(r^2 + a^2)} & 0 & 0 & 0 \\ 0 & \frac{Mr \rho^2}{\Delta(r^2 + a^2)} & 0 & 0 \\ 0 & 0 & -\frac{Mr \rho^2}{r^2 + a^2} & 0 \\ 0 & 0 & 0 & -Mr \sin^2 \theta \end{pmatrix}. \quad (33)$$

The above first matrix is just ‘exact Kerr solution’. While the second matrix can be seen as deviation or deformity from ‘exact solution’. Then, it makes sense to see this deviation as approximate solution of

$$g_{\mu\nu} = g_{\mu\nu}^{(0)} + h_{\mu\nu}, \quad (34)$$

where $g_{\mu\nu}^{(0)}$ is some known exact solution (that in this case is Kerr solution) and $h_{\mu\nu}$ is the perturbation. For Einstein’s vacuum equation it is possible to obtain explicitly the vacuum perturbation equations from an arbitrary exact solution (Wald 1984).

Gravitational wave observations of extreme-mass-ratio-inspirals (EMRIs) by LISA will provide unique evidence for the identity of the supermassive objects in galactic nuclei. It is commonly assumed that these objects are indeed Kerr black holes. K. Glampedakis and S. Babak argue that from the observed signal, LISA will have the potential to prove (or disprove) this assumption, by extracting the first few multipole moments of the spacetime outside these objects. The possibility of discovering a non-Kerr object should be taken into account when constructing waveform templates for LISA’s data analysis tools. They provide a prescription for building a ‘quasi-Kerr’ metric, that is a metric that slightly deviates from Kerr, and present results on how this deviation impacts orbital motion and the emitted waveform (Glampedakis & Babak 2006).

4.4 Deformed Schwarzschild spacetime

Another example can be given in spherical coordinates (t, r, θ, ϕ) where the components of metric tensor of Minkowski flat spacetime is:

$$\eta_{\mu\nu} = \text{diag}(1, -1, -r^2, -r^2 \sin^2 \theta),$$

with the respective inverse

$$\eta^{\mu\nu} = \text{diag}(1, -1, -\frac{1}{r^2}, -\frac{1}{r^2 \sin^2 \theta}).$$

As we have seen before, let us describe a simpler (quasi-)idempotent tensor in spherical coordinates given by

$$\Upsilon_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & r^2 & -r^2 \sin \theta \\ 0 & 0 & -r^2 \sin \theta & r^2 \sin^2 \theta \end{pmatrix} \quad (35)$$

with

$$\Upsilon_{\mu}^{\nu} = \Upsilon_{\mu\alpha} \eta^{\alpha\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -1 & \frac{1}{\sin \theta} \\ 0 & 0 & \sin \theta & -1 \end{pmatrix} \quad (36)$$

and

$$\Upsilon^{\mu}_{\nu} = \eta^{\mu\alpha} \Upsilon_{\alpha\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -1 & \sin \theta \\ 0 & 0 & \frac{1}{\sin \theta} & -1 \end{pmatrix}. \quad (37)$$

¹In this paragraph we assume $G = 1$ and $c = 1$.

Then $\text{Tr}(\Upsilon) = \Upsilon^\mu{}_\mu = -4$. One can show that $\Upsilon^{\mu\nu}$ given by $\eta^{\mu\alpha}\Upsilon_{\alpha\beta}\eta^{\beta\nu}$ is,

$$\Upsilon^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & -\frac{1}{r^2 \sin \theta} \\ 0 & 0 & -\frac{1}{r^2 \sin \theta} & \frac{1}{r^2 \sin^2 \theta} \end{pmatrix} \quad (38)$$

that satisfies the algebraic relation (4),

$$\begin{aligned} \Upsilon_{\mu\alpha}\Upsilon^{\alpha\nu} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & -\frac{2}{\sin \theta} \\ 0 & 0 & -2 \sin \theta & \frac{2}{\sin \theta} \end{pmatrix} \\ &= -2 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -1 & \frac{1}{\sin \theta} \\ 0 & 0 & \sin \theta & -1 \end{pmatrix} = -2\Upsilon_\mu{}^\nu. \end{aligned} \quad (39)$$

From expression (11) we have the metric field $g_{\mu\nu} = e^\Phi \eta_{\mu\nu} + \sinh \Phi \Upsilon_{\mu\nu}$, it follows that,

$$\begin{aligned} g_{00} &= e^\Phi, \\ g_{rr} &= -\frac{1}{e^\Phi}, \\ g_{\theta\theta} &= -r^2 \cosh \Phi, \\ g_{\phi\phi} &= -r^2 \sin^2 \theta \cosh \Phi, \\ g_{\theta\phi} &= g_{\phi\theta} = -r^2 \sin \theta \sinh \Phi, \end{aligned} \quad (40)$$

in the case that $\Phi = -\frac{2GM}{c^2 r} \ll 1$ with line element expanded in the first order of the strength of gravity,

$$\begin{aligned} ds^2 &= \underbrace{\left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 - \frac{dr^2}{\left(1 - \frac{2GM}{c^2 r}\right)} - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2}_{\text{Schwarzschild spacetime}} \\ &+ 4 \left(\frac{GM}{c^2}\right) r \sin \theta d\theta d\phi, \end{aligned} \quad (41)$$

This metric field is asymptotically flat, the components approach those of Minkowski spacetime in spherical coordinates. If only $g_{\theta\phi}$ would be vanished, we could obtain the Schwarzschild spacetime. However, it is necessary the terms $\Upsilon_{\theta\phi} = \Upsilon_{\phi\theta} = -r^2 \sin \theta$ in the symmetric tensor of equation (35) for this tensor may obey the algebraic relation (4). Nevertheless, it raises a question: what kind of gravitating source should distort a spherically symmetric spacetime? The main purpose of this section is to illustrate some (quasi-)idempotent tensors Υ that must satisfy the algebraic relation (4). There are many works with discussions about Yilmaz metric field and their sources. However, this paper does not discuss the physics of gravitating sources of circularly polarized wave, rotating bodies and deformed Schwarzschild spacetime purposed here. In forthcoming work, it will be necessary to analyse principally the gravitating source that distort the static Schwarzschild solution.

These examples suggest that the $\text{Tr}(\Upsilon) \in \mathbb{Z}$ are constant numbers. Forward it will be necessary to calculate the derivative of $\text{Tr}(\Upsilon)$ in any situations, thus we shall assume that,

$$\partial_\alpha \text{Tr}(\Upsilon) = 0 \quad (42)$$

in this paper.

5 Adjoint Metric Field, Christoffel Symbols and determinant

5.1 Adjoint Metric Field

One can see that spacetime (11) is asymptotically flat. The Minkowski spacetime is the universal covering space for all such derived spacetimes, and in this sense it is possible to restore Minkowski spacetime in (11) if one turns off the gravitational strength in metric field, or in other words, if $\Phi = 0$ then $g_{\mu\nu} = \eta_{\mu\nu}$. However one can construct a kind of spacetime with hyperbolic cosine instead of hyperbolic sine that is not asymptotically flat,

$$\check{g}_{\mu\nu} = e^\Phi \eta_{\mu\nu} + \cosh(\Phi) \Upsilon_{\mu\nu}, \quad (43)$$

with respective inverse,

$$\check{g}^{\mu\nu} = e^{-\Phi} \eta^{\mu\nu} + \cosh(\Phi) \Upsilon^{\mu\nu}, \quad (44)$$

and with $\check{g}_{\mu\nu} \check{g}^{\nu\alpha} = \delta_\mu{}^\alpha$ since $\Upsilon_{\mu\nu} \Upsilon^{\nu\alpha} = -2\Upsilon_\mu{}^\alpha$ from equation (4). A map between metrics field (11) and (43) is obtained through partial derivative in Φ :

$$\check{g}_{\mu\nu} = \frac{\partial g_{\mu\nu}}{\partial \Phi} \quad \text{and} \quad \check{g}^{\mu\nu} = -\frac{\partial g^{\mu\nu}}{\partial \Phi}. \quad (45)$$

also one has $g_{\mu\nu} = \frac{\partial^2 g_{\mu\nu}}{\partial \Phi^2}$ and $g^{\mu\nu} = \frac{\partial^2 g^{\mu\nu}}{\partial \Phi^2}$.

This ‘adjoint metric field’ (43) helps us to simplify the Christoffel symbols. But before, it is necessary to establish some relationships between \mathbf{g} and $\check{\mathbf{g}}$,

$$\begin{aligned} g_{\mu\nu} \check{g}^{\nu\alpha} &= (e^\Phi \eta_{\mu\nu} + \sinh(\Phi) \Upsilon_{\mu\nu})(e^{-\Phi} \eta^{\nu\alpha} + \cosh(\Phi) \Upsilon^{\nu\alpha}) \\ &= \delta_\mu{}^\alpha + [e^\Phi \cosh(\Phi) + e^{-\Phi} \sinh(\Phi)] \Upsilon_\mu{}^\alpha \\ &+ \sinh(\Phi) \cosh(\Phi) \Upsilon_{\mu\nu} \Upsilon^{\nu\alpha}, \end{aligned} \quad (46)$$

with $\Upsilon_{\mu\nu} \Upsilon^{\nu\alpha} = -2\Upsilon_\mu{}^\alpha$ and hyperbolic expressions $e^\Phi = \cosh(\Phi) + \sinh(\Phi)$ and $e^{-\Phi} = \cosh(\Phi) - \sinh(\Phi)$ we have:

$$g_{\mu\nu} \check{g}^{\nu\alpha} = \delta_\mu{}^\alpha + \Upsilon_\mu{}^\alpha, \quad (47)$$

and,

$$g^{\mu\nu} \check{g}_{\nu\alpha} = \delta^\mu{}_\alpha + \Upsilon^\mu{}_\alpha. \quad (48)$$

It is important to observe that:

$$g_{\mu\nu}\check{g}^{\mu\nu} = g^{\mu\nu}\check{g}_{\mu\nu} = \mathbf{Tr}(\boldsymbol{\eta}) + \mathbf{Tr}(\boldsymbol{\Upsilon}). \quad (49)$$

Further we have:

$$g_{\mu\nu}\Upsilon^{\nu\alpha} = (e^\Phi\eta_{\mu\nu} + \sinh(\Phi)\Upsilon_{\mu\nu})\Upsilon^{\nu\alpha} = e^{-\Phi}\Upsilon_\mu{}^\alpha, \quad (50)$$

$$g^{\mu\nu}\Upsilon_{\nu\alpha} = e^\Phi\Upsilon^\mu{}_\alpha, \quad (51)$$

$$\check{g}_{\mu\nu}\Upsilon^{\nu\alpha} = -e^{-\Phi}\Upsilon_\mu{}^\alpha, \quad (52)$$

$$\check{g}^{\mu\nu}\Upsilon_{\nu\alpha} = -e^\Phi\Upsilon^\mu{}_\alpha. \quad (53)$$

Nevertheless for this ‘adjoint metric field’, when there is not gravitational strength ($\Phi = 0$) it is not possible to reorder Minkowski spacetime. Because if $\Phi = 0$ then $g_{\mu\nu} = \eta_{\mu\nu} + \Upsilon_{\mu\nu}$. Unless all $\Upsilon_{\mu\nu}$ are vanished, one can not obtain Minkowski spacetime. Moreover, with $\Upsilon_{\mu\nu} \neq 0$ a spacetime (\mathcal{M}, \check{g}) has a different topology of physical spacetimes. For example, if one chooses $\Upsilon_{\mu\nu}$ from equation (17), the correspondent line element is

$$ds^2 = e^\Phi c^2 dt^2 + e^{-\Phi}(dx^2 + dy^2 + dz^2), \quad (54)$$

that has signature $(++++)$.

5.2 Christoffel Symbols

We want a manifold \mathcal{M} with Levi-Civita connection, since the connection coefficient $\boldsymbol{\Gamma}$ is given by

$$\Gamma_{\mu\nu}^\beta = \frac{1}{2}g^{\alpha\beta}(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu}). \quad (55)$$

We have that,

$$\partial_\alpha g_{\mu\nu} = (e^\Phi\eta_{\mu\nu} + \cosh(\Phi)\Upsilon_{\mu\nu})\partial_\alpha\Phi + \sinh(\Phi)\partial_\alpha\Upsilon_{\mu\nu}, \quad (56)$$

that in term of (43) is

$$\partial_\alpha g_{\mu\nu} = \check{g}_{\mu\nu}\partial_\alpha\Phi + \sinh(\Phi)\partial_\alpha\Upsilon_{\mu\nu}, \quad (57)$$

from which we obtain that Christoffel symbols are

$$\begin{aligned} \Gamma_{\mu\nu}^\beta &= \frac{1}{2}g^{\alpha\beta}[\check{g}_{\alpha\nu}\partial_\mu\Phi + \check{g}_{\alpha\mu}\partial_\nu\Phi - \check{g}_{\mu\nu}\partial_\alpha\Phi \\ &+ \sinh(\Phi)(\partial_\mu\Upsilon_{\alpha\nu} + \partial_\nu\Upsilon_{\alpha\mu} - \partial_\alpha\Upsilon_{\mu\nu})] \\ &= \frac{1}{2}[(\delta^\beta{}_\nu + \Upsilon^\beta{}_\nu)\partial_\mu\Phi + (\delta^\beta{}_\mu + \Upsilon^\beta{}_\mu)\partial_\nu\Phi - g^{\alpha\beta}\check{g}_{\mu\nu}\partial_\alpha\Phi] \\ &+ \frac{1}{2}\sinh(\Phi)g^{\alpha\beta}(\partial_\mu\Upsilon_{\alpha\nu} + \partial_\nu\Upsilon_{\alpha\mu} - \partial_\alpha\Upsilon_{\mu\nu}). \end{aligned} \quad (58)$$

Let us consider Christoffel symbols as a combination of two terms: the first is dependent of $\partial_\alpha\Phi$ and the second is dependent of $\partial_\alpha\Upsilon$,

$$\Gamma_{\mu\nu}^\beta = \Gamma_{\mu\nu}^{\beta(1)} + \Gamma_{\mu\nu}^{\beta(2)} \quad (59)$$

where,

$$\Gamma_{\mu\nu}^{\beta(1)} = \frac{1}{2}(\delta^\beta{}_\nu\delta^\alpha{}_\mu + \delta^\beta{}_\mu\delta^\alpha{}_\nu + \delta^\alpha{}_\mu\Upsilon^\beta{}_\nu + \delta^\alpha{}_\nu\Upsilon^\beta{}_\mu - g^{\alpha\beta}\check{g}_{\mu\nu})\partial_\alpha\Phi \quad (60)$$

and

$$\Gamma_{\mu\nu}^{\beta(2)} = \frac{1}{2}\sinh(\Phi)g^{\alpha\beta}(\partial_\mu\Upsilon_{\alpha\nu} + \partial_\nu\Upsilon_{\alpha\mu} - \partial_\alpha\Upsilon_{\mu\nu}). \quad (61)$$

Since $\boldsymbol{\Gamma}$ is not a tensor, it can not have intrinsic geometrical meaning as measures of how much a manifold is curved. Below we shall compute intrinsic objects as Ricci tensor and scalar curvature.

5.3 Determinant g

From identity (Schutz 1990),

$$\Gamma_{\mu\nu}^\nu = \partial_\mu(\ln \sqrt{-g}), \quad (62)$$

where $g = \det(g_{\mu\nu})$, it is possible to obtain the determinant g by contracting the indices in (59) with $\beta = \nu$ and the equation (48):

$$\begin{aligned} \Gamma_{\mu\nu}^\nu &= \frac{1}{2}[\mathbf{Tr}(\boldsymbol{\eta})\partial_\mu\Phi + \partial_\mu\Phi + \mathbf{Tr}(\boldsymbol{\Upsilon})\partial_\mu\Phi + \Upsilon^\alpha{}_\mu\partial_\alpha\Phi \\ &- (\delta^\alpha{}_\mu + \Upsilon^\alpha{}_\mu)\partial_\alpha\Phi] \\ &+ \frac{1}{2}\sinh(\Phi)g^{\alpha\nu}(\partial_\mu\Upsilon_{\alpha\nu} + \partial_\nu\Upsilon_{\alpha\mu} - \partial_\alpha\Upsilon_{\mu\nu}) \\ &= \frac{1}{2}[\mathbf{Tr}(\boldsymbol{\eta}) + \mathbf{Tr}(\boldsymbol{\Upsilon})]\partial_\mu\Phi + \frac{1}{2}\sinh(\Phi)g^{\alpha\nu}\partial_\mu\Upsilon_{\alpha\nu}, \end{aligned}$$

we note that the second term with $g^{\alpha\nu}\partial_\mu\Upsilon_{\alpha\nu}$ is vanished,

$$\begin{aligned} g^{\alpha\nu}\partial_\mu\Upsilon_{\alpha\nu} &= e^{-\Phi}\eta^{\alpha\nu}\partial_\mu\Upsilon_{\alpha\nu} - \sinh(\Phi)\Upsilon^{\alpha\nu}\partial_\mu\Upsilon_{\alpha\nu} \\ &= e^{-\Phi}\partial_\mu\mathbf{Tr}(\boldsymbol{\Upsilon}) - \sinh(\Phi)[-\partial_\mu\mathbf{Tr}(\boldsymbol{\Upsilon})] = 0, \end{aligned}$$

where we used (8) and (42), thus,

$$\Gamma_{\mu\nu}^\nu = \frac{1}{2}[\mathbf{Tr}(\boldsymbol{\eta}) + \mathbf{Tr}(\boldsymbol{\Upsilon})]\partial_\mu\Phi = \partial_\mu(\ln \sqrt{-g}). \quad (63)$$

and we finally find,

$$\sqrt{-g} = \exp\left\{\frac{\Phi}{2}[\mathbf{Tr}(\boldsymbol{\eta}) + \mathbf{Tr}(\boldsymbol{\Upsilon})]\right\}. \quad (64)$$

In fact, since $\ln(\det g_{\mu\nu}) = \mathbf{Tr}(\ln g_{\mu\nu})$. Also, one should use the identity $dg = g g^{\mu\nu}dg_{\mu\nu}$ and to compute $\sqrt{-g}$. One can verify the examples of above section in coordinates (t, x, y, z) :

Yilmaz metric:	$g = -\exp\{\Phi[4 + (-6)]\} = -e^{-2\Phi}$
circularly polarized wave:	$g = -\exp\{\Phi[4 + (-4)]\} = -1$
rotating bodies:	$g = -\exp\{\Phi[4 + (-2)]\} = -e^{2\Phi}$

The natural volume element of manifold \mathcal{M} with metric tensor (11), $dv = \sqrt{-g}d^4x$, is invariant under

coordinate transformation. An action with Lagrangian \mathcal{L} in spacetime $(\mathcal{M}, \mathbf{g})$ is given by,

$$S = \int_{\mathcal{M}} \mathcal{L} \sqrt{-g} d^4x, \quad (65)$$

then we have,

$$S = \int_{\mathcal{M}} \mathcal{L} \exp \left\{ \frac{\Phi}{2} [\mathbf{Tr}(\boldsymbol{\eta}) + \mathbf{Tr}(\boldsymbol{\Upsilon})] \right\} d^4x. \quad (66)$$

6 Newtonian Limit and Gravitational Waves

6.1 Newtonian Limit

In the Newtonian limit one assumes that velocities are small, $\left| \frac{v}{c} \right| \ll 1$, that gravitational potentials are near their Minkowski values,² and that pressures or other mechanical stresses are negligible compared to the energy densities $|P| \ll \rho c^2$.

The description of Einstein's field equation (2) will use values from Christoffel symbols from equation (59), where this has two terms: $\Gamma_{\mu\nu}^{\beta(1)}$ dependent of $\partial_\alpha \Phi$ and $\Gamma_{\mu\nu}^{\beta(2)}$ dependent of $\partial_\alpha \boldsymbol{\Upsilon}$. In this section we shall verify the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$, only in the case that $\partial_\alpha \boldsymbol{\Upsilon} = 0$, then $\Gamma_{\mu\nu}^{\beta(2)} = 0$, and in any coordinates system, the Ricci tensor is given by

$$R_{\mu\nu}^{(1)} = \partial_\alpha \Gamma_{\mu\nu}^{\alpha(1)} - \Gamma_{\mu\beta}^{\alpha(1)} \Gamma_{\nu\alpha}^{\beta(1)} - \partial_\nu \Gamma_{\mu\alpha}^{\alpha(1)} + \Gamma_{\alpha\beta}^{\alpha(1)} \Gamma_{\mu\nu}^{\beta(1)}. \quad (67)$$

Since $\partial_\alpha \boldsymbol{\Upsilon} = 0$ the first term of $R_{\mu\nu}^{(1)}$ is given by

$$\begin{aligned} \partial_\alpha \Gamma_{\mu\nu}^{\alpha(1)} &= \frac{1}{2} (\check{g}_{\mu\nu} \check{g}^{\alpha\beta} - g_{\mu\nu} g^{\alpha\beta}) \partial_\alpha \Phi \partial_\beta \Phi \\ &+ \frac{1}{2} (2\partial_\mu \partial_\nu \Phi + \Upsilon^\beta{}_\nu \partial_\mu \partial_\beta \Phi + \Upsilon^\beta{}_\mu \partial_\nu \partial_\beta \Phi \\ &- \check{g}_{\mu\nu} g^{\alpha\beta} \partial_\alpha \partial_\beta \Phi), \end{aligned} \quad (68)$$

the second term of Ricci tensor is:

$$\begin{aligned} \Gamma_{\mu\beta}^{\alpha(1)} \Gamma_{\nu\alpha}^{\beta(1)} &= \frac{3}{2} \partial_\mu \Phi \partial_\nu \Phi + \frac{1}{2} (\partial_\mu \Phi \Upsilon^\gamma{}_\nu + \partial_\nu \Phi \Upsilon^\gamma{}_\mu) \partial_\gamma \Phi \\ &- \frac{1}{2} g_{\mu\nu} g^{\lambda\gamma} \partial_\gamma \Phi \partial_\lambda \Phi + \frac{1}{2} \Upsilon^\lambda{}_\mu \Upsilon^\gamma{}_\nu \partial_\gamma \Phi \partial_\lambda \Phi, \end{aligned}$$

with (63) we can obtain the third term:

$$\partial_\nu \Gamma_{\mu\alpha}^{\alpha(1)} = \frac{1}{2} [\mathbf{Tr}(\boldsymbol{\eta}) + \mathbf{Tr}(\boldsymbol{\Upsilon})] \partial_\mu \partial_\nu \Phi, \quad (69)$$

and the fourth is

$$\begin{aligned} \Gamma_{\beta\alpha}^{\alpha(1)} \Gamma_{\mu\nu}^{\beta(1)} &= \frac{1}{4} [\mathbf{Tr}(\boldsymbol{\eta}) + \mathbf{Tr}(\boldsymbol{\Upsilon})] (2\partial_\mu \Phi \partial_\nu \Phi \\ &+ \partial_\mu \Phi \Upsilon^\beta{}_\nu \partial_\beta \Phi + \partial_\nu \Phi \Upsilon^\beta{}_\mu \partial_\beta \Phi - \check{g}_{\mu\nu} g^{\alpha\beta} \partial_\alpha \Phi \partial_\beta \Phi). \end{aligned} \quad (70)$$

²Observe that $g_{\mu\nu} = e^\Phi \eta_{\mu\nu} + \sinh(\Phi) \Upsilon_{\mu\nu} \approx (1+\Phi) \eta_{\mu\nu} + \Phi \Upsilon_{\mu\nu} = \eta_{\mu\nu} + \Phi(\eta_{\mu\nu} + \Upsilon_{\mu\nu}) = \eta_{\mu\nu} + h_{\mu\nu}$ and $g^{\mu\nu} \approx \eta^{\mu\nu} - \Phi(\eta^{\mu\nu} + \Upsilon^{\mu\nu}) = \eta^{\mu\nu} - h^{\mu\nu}$.

For Newtonian limit, we must retain only the terms in the first order in the strength of gravity Φ , or in other words, we have $\partial\Phi\partial\Phi \ll \partial\partial\Phi$ and $\Phi\partial\partial\Phi \ll \partial\partial\Phi$, with this assumption we have the second term of (68) and (69) such as,

$$\begin{aligned} R_{\mu\nu}^{(1)} &= \frac{1}{2} (2\partial_\mu \partial_\nu \Phi + \Upsilon^\beta{}_\nu \partial_\mu \partial_\beta \Phi + \Upsilon^\beta{}_\mu \partial_\nu \partial_\beta \Phi \\ &- \check{g}_{\mu\nu} g^{\alpha\beta} \partial_\alpha \partial_\beta \Phi) - \frac{1}{2} [\mathbf{Tr}(\boldsymbol{\eta}) + \mathbf{Tr}(\boldsymbol{\Upsilon})] \partial_\mu \partial_\nu \Phi. \end{aligned} \quad (71)$$

One can choose the simplest $\boldsymbol{\Upsilon}$, and it should be from (17) such as $\mathbf{Tr}(\boldsymbol{\Upsilon}) = -6$, where the Ricci tensor is now given by

$$R_{\mu\nu}^{(1)} = 2\partial_\mu \partial_\nu \Phi + \frac{1}{2} \Upsilon^\beta{}_\nu \partial_\mu \partial_\beta \Phi + \frac{1}{2} \Upsilon^\beta{}_\mu \partial_\nu \partial_\beta \Phi - \frac{1}{2} \check{g}_{\mu\nu} g^{\alpha\beta} \partial_\alpha \partial_\beta \Phi. \quad (72)$$

The scalar curvature is obtained by further contracting indices $R^{(1)} = g^{\mu\nu} R_{\mu\nu}^{(1)}$,

$$R^{(1)} = 3g^{\mu\nu} \partial_\mu \partial_\nu \Phi + g^{\mu\nu} \Upsilon^\beta{}_\nu \partial_\mu \partial_\beta \Phi, \quad (73)$$

$g^{\mu\nu} = (1 - \Phi)\eta^{\mu\nu} - \Phi\Upsilon^{\mu\nu}$ in terms of the first order in Φ , we find that the scalar curvature is

$$R^{(1)} = 3\eta^{\mu\nu} \partial_\mu \partial_\nu \Phi + \eta^{\mu\nu} \Upsilon^\beta{}_\nu \partial_\mu \partial_\beta \Phi. \quad (74)$$

Since the Newtonian limit approach is in the background Minkowski spacetime, we find that

$$R^{(1)} = 3\Box\Phi + \Upsilon^{\beta\mu} \partial_\mu \partial_\beta \Phi. \quad (75)$$

From the field equations (2) we have

$$R = -\frac{8\pi G}{c^4} T, \quad (76)$$

in the situations where Newtonian theory can be applicable $T \approx \rho c^2$. Now we can combine equations (76) and (75) such as

$$3\Box\Phi + \Upsilon^{\beta\mu} \partial_\mu \partial_\beta \Phi = -\frac{8\pi G}{c^2} \rho. \quad (77)$$

We assume that velocities are small, then

$$\Box = \frac{\partial^2}{c^2 \partial t^2} - \nabla^2 \approx -\nabla^2. \quad (78)$$

And in particular from (17) we have $\Upsilon_{\mu\nu} \Rightarrow 2\delta_{ij}$, where δ_{ij} is the Kronecker delta in \mathbb{R}^3 . The field equation (77) results in:

$$-3\nabla^2\Phi + 2\delta^{ij} \partial_i \partial_j \Phi = -\frac{8\pi G}{c^2} \rho, \quad (79)$$

or

$$\nabla^2\Phi = \frac{8\pi G}{c^2} \rho. \quad (80)$$

We can introduce $\Phi = \frac{2\varphi_N}{c^2}$, where φ_N is the Newtonian potential. Thus we can obtain the Poisson's equation of Newton's law of gravitation

$$\nabla^2 \varphi_N = 4\pi G\rho, \quad (81)$$

in this way for a point source with mass M we have

$$\varphi_N = -\frac{GM}{r}. \quad (82)$$

Consequently $\Phi = -\frac{2GM}{c^2 r}$. In this discussion that follows (71) we assumed that $\Upsilon_{\mu\nu}$ is from (17), it results that the line element is given by (20) at the first order in Φ .

6.2 Gravitational Waves

Gravitational waves are one of the most important physical phenomena associated with the presence of strong and dynamic gravitational fields. Though such gravitational radiation has not yet been detected directly. There is strong indirect evidence for its existence around the famous binary pulsar PSR 1913+16 (Taylor & Weisberg 1982), that in 1974 it was discovered by R.A. Hulse and J.H. Taylor (Hulse & Taylor 1975), a discovery for which they were awarded the 1993 Nobel Prize.

Here, for a description of plane gravitational waves we can assume that Φ is constant so $\partial_\alpha \Phi = 0$, implying in $\Gamma_{\mu\nu}^{\beta(1)} = 0$ in (59), thus we have $\Gamma_{\mu\nu}^\beta = \Gamma_{\mu\nu}^{\beta(2)}$ given by equation (61):

$$\Gamma_{\mu\nu}^{\beta(2)} = \frac{1}{2} \sinh(\Phi) g^{\alpha\beta} (\partial_\mu \Upsilon_{\alpha\nu} + \partial_\nu \Upsilon_{\alpha\mu} - \partial_\alpha \Upsilon_{\mu\nu}). \quad (83)$$

One can obtain the field equation writing the Ricci tensor in terms of Christoffel symbols above,

$$R_{\mu\nu}^{(2)} = \partial_\alpha \Gamma_{\mu\nu}^{\alpha(2)} - \Gamma_{\mu\beta}^{\alpha(2)} \Gamma_{\nu\alpha}^{\beta(2)} - \partial_\nu \Gamma_{\mu\alpha}^{\alpha(2)} + \Gamma_{\alpha\beta}^{\alpha(2)} \Gamma_{\mu\nu}^{\beta(2)}. \quad (84)$$

From equation (63) we have

$$\Gamma_{\alpha\nu}^\alpha = \frac{1}{2} [\text{Tr}(\boldsymbol{\eta}) + \text{Tr}(\boldsymbol{\Upsilon})] \partial_\nu \Phi = 0, \text{ such as,}$$

$$R_{\mu\nu}^{(2)} = \partial_\alpha \Gamma_{\mu\nu}^{\alpha(2)} - \Gamma_{\mu\beta}^{\alpha(2)} \Gamma_{\nu\alpha}^{\beta(2)}, \quad (85)$$

As shown in appendix, the The above Ricci tensor is given by,

$$\begin{aligned} R_{\mu\nu}^{(2)} = & \frac{1}{2} \sinh^2(\Phi) \partial_\alpha \Upsilon^{\alpha\beta} (\partial_\mu \Upsilon_{\beta\nu} + \partial_\nu \Upsilon_{\beta\mu} - \partial_\beta \Upsilon_{\mu\nu}) \\ & + \frac{1}{2} \sinh(\Phi) g^{\alpha\beta} (\partial_\alpha \partial_\mu \Upsilon_{\beta\nu} + \partial_\alpha \partial_\nu \Upsilon_{\beta\mu} - \partial_\alpha \partial_\beta \Upsilon_{\mu\nu}) \\ & - \frac{1}{4} \sinh^2(\Phi) [g^{\alpha\gamma} g^{\beta\lambda} \partial_\mu \Upsilon_{\gamma\beta} \partial_\nu \Upsilon_{\alpha\lambda} \\ & + 2(g^{\alpha\gamma} g^{\beta\lambda} - g^{\alpha\beta} g^{\gamma\lambda}) \partial_\beta \Upsilon_{\gamma\mu} \partial_\alpha \Upsilon_{\nu\lambda}]. \end{aligned} \quad (86)$$

The study of gravitational waves involves essentially the approximation of Einstein's weak field equation, from these results one can obtain wave equation with low amplitudes, $\Phi \ll 1$. This can only be brought in the first order of the gravitational potential to find,

$$R_{\mu\nu}^{(2)} = \frac{1}{2} \sinh(\Phi) g^{\alpha\beta} (\partial_\alpha \partial_\mu \Upsilon_{\beta\nu} + \partial_\alpha \partial_\nu \Upsilon_{\beta\mu} - \partial_\alpha \partial_\beta \Upsilon_{\mu\nu}), \quad (87)$$

with $\sinh(\Phi) g^{\alpha\beta} \approx \Phi[(1-\Phi)\eta^{\alpha\beta} - \Phi\Upsilon^{\alpha\beta}] = \Phi\eta^{\alpha\beta}$, one can obtain the Ricci tensor in Minkowski background spacetime, which this yields,

$$R_{\mu\nu}^{(2)} = \frac{1}{2} \Phi (\partial^\beta \partial_\mu \Upsilon_{\beta\nu} + \partial^\beta \partial_\nu \Upsilon_{\beta\mu} - \square \Upsilon_{\mu\nu}), \quad (88)$$

The gauge choice $\partial^\beta \Upsilon_{\beta\nu} = 0$, simplifies Ricci tensor in:

$$R_{\mu\nu}^{(2)} = -\frac{1}{2} \Phi \square \Upsilon_{\mu\nu}. \quad (89)$$

One should observe that scalar curvature

$$R^{(2)} = g^{\mu\nu} R_{\mu\nu}^{(2)}$$

is vanished for the first order in Φ , because

$$R^{(2)} = \frac{1}{2} \eta^{\mu\nu} \Phi \square \Upsilon_{\mu\nu} = \frac{1}{2} \square \text{Tr}(\boldsymbol{\Upsilon}) = 0,$$

since the trace of tensor $\boldsymbol{\Upsilon}$ is a constant number, in accordance with equation (42). Then, for the field equation (2) the gravitational wave equation obtained in the vacuum is only,

$$\square \Upsilon_{\mu\nu} = 0. \quad (90)$$

It is necessary that the tensor $\Upsilon_{\mu\nu}$ satisfies the condition (4) and the above wave equation. Its shape should be, for example, the tensor (22) with $\zeta = \kappa_\mu x^\mu = \omega t - kz$, a solution for gravitational plane waves with circular polarization traveling in the z direction, with amplitude Φ .

7 Conclusion

This paper deals with a tensorial structure that assumes a (quasi-)idempotent feature to be able to improve at least the linear tensorial template of some tensor metric fields. It is clear that Einstein's field equations are non-linear, however, with these (quasi-)idempotent tensorial structure, without quadratic tensorial values, the non-linearity becomes more moderate although there is a price to pay. The part that carries the dynamical information, the strength of gravity is tied to the tensorial

structure by exponential functions. In this approach the metric field can be characterized by a background spacetime conformally flat affected by a disturbance. We have approached some examples in this tensorial structure that results in exponential metric fields, we can point out as the main exponential metric obtained in this paper which has been extensively explored: the Yilmaz exponential metric (Yilmaz 1958, 1992, 1976, 1982, 1973, 1977, 1997; Clapp 1973; Robertson 1999a,b; Ibison 2006). H. Yilmaz has argued that in his theory, the gravitational field can be quantized via Feynman's method (Yilmaz 1995; Alley 1995). Further, it has been found that the quantized theory is finite. Incidentally in the exponential metric fields approached in this work just as in the Yilmaz theory there are no black holes in the sense of event horizons, but there can be stellar collapse (Robertson 1999a,b). However, there are no point singularities.

Interesting results obtained in this work from exponential metric fields are: circularly polarized wave; rotating bodies that in the first order is a deformation of Kerr metric and also we have a deformed static spherically symmetric spacetime. Many discussions around massive stellar objects have suggested, for example, that Kerr metric should be slightly deviated from Kerr. The possibility of discovering a non-Kerr object should be taken into account when constructing waveform templates for LISA's data analysis tools (Glampedakis & Babak 2006; White 2006). The technological development is ripe enough so much so in the years to come we might be able to test the second order relativist-gravity effects and may lead to answers to some important questions of gravity.

In this work, we have obtained a simple and general expression for the volume element of a manifold in coordinates (t, x, y, z) given in terms of strength of gravity and of traces of tensors η and Υ . It is possible that an analysis of any Lagrangian of field interacting with gravity will become easier. An interesting observation is the spacetime of circularly polarized plane wave, in this spacetime the volume element $\sqrt{-g} d^4x$ is the same of Minkowski spacetime, in this sense this gravitational radiation obtained from exponential metric field does not modify the volume element of background Minkowski spacetime where this plane wave travels onto. Moreover, it was purposed and verified the Newtonian limit as solution for Einstein's equation, since we can assume that the trace of stress-energy tensor is $T \approx \rho c^2$. Other important solution of Einstein's equation analysed in this paper was the plane gravitational wave for the empty space since we have considered the vanished stress-energy tensor to the first order in Φ . Both solved Einstein's equations

for Newtonian limit and plane gravitational wave propagating in the vacuum are cases that the strength of gravity is small, $\Phi \ll 1$. We have analysed the Newtonian limit in the case that $\partial_\alpha \Upsilon = 0$, and analysed the plane gravitational wave considering the strength of gravity as a constant term, thus we had two independent Ricci tensors, $R_{\mu\nu}^{(1)}$ which $\partial_\alpha \Upsilon = 0$ (for Newtonian limit) and $R_{\mu\nu}^{(2)}$ which $\partial_\alpha \Phi = 0$ (for plane gravitational wave). In a forthcoming work, an analysis of Einstein's equations with both non-vanished $\partial_\alpha \Upsilon$ and $\partial_\alpha \Phi$, will be considered.

It is missing a discussion about quantities of physical interest in the solutions of Einstein's equations which describe the exterior and interior gravitational field. Yilmaz has argued the existence of the matter part in the right-hand side of the field equations correspondent to field energy in the exterior. This paper lacks a discussion about the interior and the exterior field energies denoted by a total stress-energy tensor. An analysis about the total stress-energy tensor will be the object of a forthcoming study, where the physical consequences of terms of deformity in Kerr and Schwarzschild solutions could be analysed.

We know that the dark energy and the dark matter problems are challenges to modern astrophysics and cosmology; as a typical example, we could mention the galactic rotation curves of spiral galaxies, that probably, indicates the possible failure of both Newtonian gravity and General Relativity on galactic and intergalactic scales. To explain astrophysical and cosmological problems with arguments against dark energy and dark matter many works have been devoted to the possibility that the Einstein-Hilbert Lagrangian, linear in the Ricci scalar R , should be generalized. In this sense, the choice of a generic function $f(R)$ can be derived by matching the data and the requirement that no exotic ingredient have to be added (Allemandi et al. 2004; Barrow et al. 1983; Capozziello 2002; Capozziello et al. 2003, 2005; Carroll et al. 2004, 2005; Faraoni 2008; Flanagan 2003; Koivisto 2006; Nojiri et al. 2007, 2003a,b; Sotiriou et al. 2010). This class of theories when linearized exhibits others polarization modes for the gravitational waves, of which two correspond to the massless graviton and others such as massive scalar and ghost modes in $f(R)$ gravity (Bellucci et al. 2009; Bogdanos et al. 2009). In this way, analyses in any order to $f(R)$ gravity with 'exponential metrics' proposed in the present work could give a positive contribution to the debate of astrophysical and cosmological questions.

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A Details of Calculation for Rotating Bodies

A.1 Multiplicative Properties of Υ in Boyer-Lindquist coordinates

The metric tensor $\eta_{\mu\nu}$ of Minkowski flat spacetime in Boyer-Lindquist coordinates is given by:

$$\eta_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{r^2+a^2(\cos(\theta))^2}{r^2+a^2} & 0 & 0 \\ 0 & 0 & -r^2 - a^2(\cos(\theta))^2 & 0 \\ 0 & 0 & 0 & -(r^2+a^2)(\sin(\theta))^2 \end{bmatrix}, \quad (\text{A1})$$

while the (quasi-)idempotent tensor $\Upsilon_{\mu\nu}$ from (25) in the same coordinates is:

$$\Upsilon_{\mu\nu} = (-2) \begin{bmatrix} (\cosh(\Lambda))^2 & 0 & 0 & \sinh(\Lambda) \cosh(\Lambda) \sqrt{r^2+a^2} \sin(\theta) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sinh(\Lambda) \cosh(\Lambda) \sqrt{r^2+a^2} \sin(\theta) & 0 & 0 & (\sinh(\Lambda))^2 (r^2+a^2) (\sin(\theta))^2 \end{bmatrix}. \quad (\text{A2})$$

Let us verify that above tensor $\Upsilon_{\mu\nu}$ satisfies the algebraic relation (4). We begin with the inverse metric tensor of Minkowski space:

$$\eta^{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{r^2+a^2}{r^2+a^2(\cos(\theta))^2} & 0 & 0 \\ 0 & 0 & -\left(r^2+a^2(\cos(\theta))^2\right)^{-1} & 0 \\ 0 & 0 & 0 & -\frac{1}{(r^2+a^2)(\sin(\theta))^2} \end{bmatrix}, \quad (\text{A3})$$

such as $\Upsilon^\mu{}_\nu$ e $\Upsilon_\mu{}^\nu$ are calculated with contracting indices,

$$\Upsilon^\mu{}_\nu = \eta^{\mu\alpha} \Upsilon_{\alpha\nu} = (-2) \begin{bmatrix} (\cosh(\Lambda))^2 & 0 & 0 & \sinh(\Lambda) \cosh(\Lambda) \sqrt{r^2+a^2} \sin(\theta) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{\sinh(\Lambda) \cosh(\Lambda)}{\sqrt{r^2+a^2} \sin(\theta)} & 0 & 0 & -(\sinh(\Lambda))^2 \end{bmatrix} \quad (\text{A4})$$

and

$$\Upsilon_\mu{}^\nu = \Upsilon_{\mu\alpha} \eta^{\alpha\nu} = (-2) \begin{bmatrix} (\cosh(\Lambda))^2 & 0 & 0 & -\frac{\sinh(\Lambda) \cosh(\Lambda)}{\sqrt{r^2+a^2} \sin(\theta)} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sinh(\Lambda) \cosh(\Lambda) \sqrt{r^2+a^2} \sin(\theta) & 0 & 0 & -(\sinh(\Lambda))^2 \end{bmatrix}. \quad (\text{A5})$$

Notice that it follows the contravariant tensor $\Upsilon^{\mu\nu}$ given by:

$$\Upsilon^{\mu\nu} = \eta^{\mu\alpha} \Upsilon_{\alpha\beta} \eta^{\beta\nu} = (-2) \begin{bmatrix} (\cosh(\Lambda))^2 & 0 & 0 & -\frac{\sinh(\Lambda) \cosh(\Lambda)}{\sqrt{r^2+a^2} \sin(\theta)} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{\sinh(\Lambda) \cosh(\Lambda)}{\sqrt{r^2+a^2} \sin(\theta)} & 0 & 0 & \frac{(\sinh(\Lambda))^2}{(r^2+a^2)(\sin(\theta))^2} \end{bmatrix}. \quad (\text{A6})$$

Finally one can obtain that $\Upsilon_{\mu\nu} \Upsilon^{\nu\alpha} = -2\Upsilon_{\mu}^{\alpha}$, this is proved by observing that,

$$\Upsilon_{\mu\nu} \Upsilon^{\nu\alpha} = (-2) \cdot (-2) \begin{bmatrix} (\cosh(\Lambda))^2 & 0 & 0 & -\frac{\sinh(\Lambda) \cosh(\Lambda)}{\sqrt{r^2+a^2} \sin(\theta)} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sinh(\Lambda) \cosh(\Lambda) \sqrt{r^2+a^2} \sin(\theta) & 0 & 0 & -(\sinh(\Lambda))^2 \end{bmatrix} = -2\Upsilon_{\mu}^{\alpha}. \quad (\text{A7})$$

It is easy to see that the above tensor is the same of equation (A5) as claimed.

A.2 Calculation of components

Let us compute the components of metric tensor $g_{\mu\nu} = e^{\Phi} \eta_{\mu\nu} + \sinh(\Phi) \Upsilon_{\mu\nu}$ for rotating bodies. First we may calculate $g_{00} = e^{\Phi} \eta_{00} + \sinh(\Phi) \Upsilon_{00} \approx (1 + \Phi) \eta_{00} + \Phi \Upsilon_{00}$,

$$g_{00} = 1 + \Phi - 2\Phi \cosh^2 \Lambda = 1 + \frac{Mr}{(r^2 + a^2)} - 2 \cdot \frac{Mr}{(r^2 + a^2)} \cdot \frac{r^2 + a^2}{\rho^2}, \quad (\text{A8})$$

since $\Phi = \frac{Mr}{(r^2 + a^2)}$ from (31), and also $\cosh \Lambda = \frac{\sqrt{r^2 + a^2}}{\rho}$ from (29),

$$g_{00} = 1 - \frac{2Mr}{\rho^2} + \frac{Mr}{(r^2 + a^2)} = \frac{\rho^2 - 2Mr}{\rho^2} + \frac{Mr}{(r^2 + a^2)}, \quad (\text{A9})$$

with $\rho = r^2 + a^2 \cos^2 \theta$ then,

$$\begin{aligned} g_{00} &= \frac{r^2 + a^2 \cos^2 \theta - 2Mr}{\rho^2} + \frac{Mr}{(r^2 + a^2)} = \frac{r^2 + a^2(1 - \sin^2 \theta) - 2Mr}{\rho^2} + \frac{Mr}{(r^2 + a^2)} \\ &= \frac{r^2 + a^2 - 2Mr - a^2 \sin^2 \theta}{\rho^2} + \frac{Mr}{(r^2 + a^2)} \\ &= \frac{\Delta - a^2 \sin^2 \theta}{\rho^2} + \frac{Mr}{(r^2 + a^2)}, \end{aligned} \quad (\text{A10})$$

where $\Delta = r^2 - 2Mr - a^2$

Calculation of g_{03} :

$$\begin{aligned} g_{03} &= -2 \sinh(\Phi) \sinh \Lambda \cosh \Lambda \sqrt{r^2 + a^2} \sin \theta \\ &= -2 \left(\frac{Mr}{r^2 + a^2} \right) \left(-\frac{a \sin \theta}{\rho} \right) \left(\frac{\sqrt{r^2 + a^2}}{\rho} \right) \sqrt{r^2 + a^2} \sin \theta = \frac{2Mra \sin^2 \theta}{\rho^2}, \end{aligned} \quad (\text{A11})$$

again we used the definition (29).

Calculation of g_{22} :

$$g_{22} = -e^\Phi \rho^2 = -\left(1 + \frac{Mr}{(r^2 + a^2)}\right) \rho^2 = -\rho^2 - \frac{Mr\rho^2}{r^2 + a^2} \quad (\text{A12})$$

Calculation of g_{33} :

$$\begin{aligned} g_{33} &= -e^\Phi (r^2 + a^2) \sin^2 \theta - 2 \sinh(\Phi) \sinh^2 \Lambda (r^2 + a^2) \sin^2 \theta \\ &= -\left(1 + \frac{Mr}{(r^2 + a^2)}\right) (r^2 + a^2) \sin^2 \theta - 2 \left(\frac{Mr}{r^2 + a^2}\right) \left(\frac{a^2 \sin^2 \theta}{\rho^2}\right) (r^2 + a^2) \sin^2 \theta \\ &= -(r^2 + a^2) \sin^2 \theta - \frac{2Mra^2 \sin^4 \theta}{\rho^2} - Mr \sin^2 \theta \\ &= -\frac{\rho^2 (r^2 + a^2) \sin^2 \theta + 2Mra^2 \sin^4 \theta}{\rho^2} - Mr \sin^2 \theta \\ &= -\frac{(r^2 + a^2 \cos^2 \theta)(r^2 + a^2) \sin^2 \theta + 2Mra^2 \sin^4 \theta}{\rho^2} - Mr \sin^2 \theta \\ &= -\frac{(r^2 + a^2 - a^2 \sin^2 \theta)(r^2 + a^2) \sin^2 \theta + 2Mra^2 \sin^4 \theta}{\rho^2} - Mr \sin^2 \theta \\ &= -\frac{[(r^2 + a^2) - a^2 \sin^2 \theta] (r^2 + a^2) \sin^2 \theta + 2Mra^2 \sin^4 \theta}{\rho^2} - Mr \sin^2 \theta \\ &= -\frac{(r^2 + a^2)^2 \sin^2 \theta + (r^2 - 2Mr + a^2) a^2 \sin^4 \theta}{\rho^2} - Mr \sin^2 \theta \\ &= \frac{-\sin^2 \theta}{\rho^2} \left[(r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta \right] - Mr \sin^2 \theta \end{aligned} \quad (\text{A13})$$

Calculation of g_{11} :

$$g_{11} = -e^\Phi \frac{\rho^2}{r^2 + a^2}, \quad (\text{A14})$$

we have that $\Delta = r^2 - 2Mr - a^2$ can be given by:

$$\begin{aligned} \Delta &= r^2 - 2Mr - a^2 = (r^2 + a^2) - 2Mr \\ \Delta &= (r^2 + a^2) \left(1 - \frac{2Mr}{(r^2 + a^2)}\right), \end{aligned} \quad (\text{A15})$$

which implies that

$$\frac{1}{r^2 + a^2} = \frac{1}{\Delta} \left(1 - \frac{2Mr}{(r^2 + a^2)}\right). \quad (\text{A16})$$

Now, the component g_{11} is given by:

$$g_{11} = -e^\Phi \frac{\rho^2}{r^2 + a^2} = -\left(1 + \frac{Mr}{(r^2 + a^2)}\right) \frac{\rho^2}{\Delta} \left(1 - \frac{2Mr}{(r^2 + a^2)}\right). \quad (\text{A17})$$

Hence we have that $\frac{Mr}{r^2 + a^2} \ll 1$, the component g_{11} is given just in the first order:

$$\begin{aligned} g_{11} &= -\frac{\rho^2}{\Delta} \left(1 - \frac{Mr}{(r^2 + a^2)}\right) \\ g_{11} &= -\frac{\rho^2}{\Delta} + \frac{Mr\rho^2}{\Delta(r^2 + a^2)}. \end{aligned} \quad (\text{A18})$$

B Calculation of Ricci tensor $R_{\mu\nu}^{(2)}$

The Ricci tensor from section VI, used to evaluated gravitational waves, is given by:

$$R_{\mu\nu}^{(2)} = \partial_\alpha \Gamma_{\mu\nu}^{\alpha(2)} - \Gamma_{\mu\beta}^{\alpha(2)} \Gamma_{\nu\alpha}^{\beta(2)} - \partial_\nu \Gamma_{\mu\alpha}^{\alpha(2)} + \Gamma_{\alpha\beta}^{\alpha(2)} \Gamma_{\mu\nu}^{\beta(2)}. \quad (\text{B1})$$

Let Φ be constant. Then $\Gamma_{\alpha\nu}^\alpha = \frac{1}{2}[\mathbf{Tr}(\boldsymbol{\eta}) + \mathbf{Tr}(\boldsymbol{\Upsilon})]\partial_\nu\Phi$ are vanished such that

$$R_{\mu\nu}^{(2)} = \partial_\alpha \Gamma_{\mu\nu}^{\alpha(2)} - \Gamma_{\mu\beta}^{\alpha(2)} \Gamma_{\nu\alpha}^{\beta(2)}. \quad (\text{B2})$$

Let us calculate the first term:

$$\partial_\alpha \Gamma_{\mu\nu}^{\alpha(2)} = \frac{1}{2} \sinh(\Phi) \partial_\alpha g^{\alpha\beta} (\partial_\mu \Upsilon_{\beta\nu} + \partial_\nu \Upsilon_{\beta\mu} - \partial_\beta \Upsilon_{\mu\nu}) + \frac{1}{2} \sinh(\Phi) g^{\alpha\beta} (\partial_\alpha \partial_\mu \Upsilon_{\beta\nu} + \partial_\alpha \partial_\nu \Upsilon_{\beta\mu} - \partial_\alpha \partial_\beta \Upsilon_{\mu\nu}). \quad (\text{B3})$$

If Φ is constant, then we have $\partial_\alpha g^{\alpha\beta} = \sinh(\Phi) \partial_\alpha \Upsilon^{\alpha\beta}$, such that,

$$\partial_\alpha \Gamma_{\mu\nu}^{\alpha(2)} = \frac{1}{2} \sinh^2(\Phi) \partial_\alpha \Upsilon^{\alpha\beta} (\partial_\mu \Upsilon_{\beta\nu} + \partial_\nu \Upsilon_{\beta\mu} - \partial_\beta \Upsilon_{\mu\nu}) + \frac{1}{2} \sinh(\Phi) g^{\alpha\beta} (\partial_\alpha \partial_\mu \Upsilon_{\beta\nu} + \partial_\alpha \partial_\nu \Upsilon_{\beta\mu} - \partial_\alpha \partial_\beta \Upsilon_{\mu\nu}). \quad (\text{B4})$$

Let us calculate the second term from Ricci tensor (B2):

$$\begin{aligned} \Gamma_{\mu\beta}^{\alpha(2)} \Gamma_{\nu\alpha}^{\beta(2)} &= \frac{1}{2} \sinh(\Phi) g^{\alpha\gamma} (\partial_\mu \Upsilon_{\gamma\beta} + \partial_\beta \Upsilon_{\gamma\mu} - \partial_\gamma \Upsilon_{\mu\beta}) \frac{1}{2} \sinh(\Phi) g^{\beta\lambda} (\partial_\alpha \Upsilon_{\nu\lambda} + \partial_\nu \Upsilon_{\alpha\lambda} - \partial_\lambda \Upsilon_{\alpha\nu}) \\ &= \frac{1}{4} \sinh^2(\Phi) g^{\alpha\gamma} g^{\beta\lambda} [(\partial_\mu \Upsilon_{\gamma\beta} \partial_\alpha \Upsilon_{\nu\lambda} - \partial_\mu \Upsilon_{\gamma\beta} \partial_\lambda \Upsilon_{\alpha\nu}) + (\partial_\beta \Upsilon_{\gamma\mu} \partial_\alpha \Upsilon_{\nu\lambda} + \partial_\gamma \Upsilon_{\mu\beta} \partial_\lambda \Upsilon_{\alpha\nu}) \\ &\quad + (\partial_\beta \Upsilon_{\gamma\mu} \partial_\nu \Upsilon_{\alpha\lambda} - \partial_\gamma \Upsilon_{\mu\beta} \partial_\nu \Upsilon_{\alpha\lambda}) - (\partial_\beta \Upsilon_{\gamma\mu} \partial_\lambda \Upsilon_{\alpha\nu} + \partial_\gamma \Upsilon_{\mu\beta} \partial_\alpha \Upsilon_{\nu\lambda}) + \partial_\mu \Upsilon_{\gamma\beta} \partial_\nu \Upsilon_{\alpha\lambda}]. \end{aligned}$$

The contracting indices α, β, γ and λ can be manipulated as following: $\alpha \leftrightarrow \lambda$ and $\beta \leftrightarrow \gamma$ in the second term of each above parenthesis, that result in

$$\Gamma_{\mu\beta}^{\alpha(2)} \Gamma_{\nu\alpha}^{\beta(2)} = \frac{1}{4} \sinh^2(\Phi) g^{\alpha\gamma} g^{\beta\lambda} [2\partial_\beta \Upsilon_{\gamma\mu} \partial_\alpha \Upsilon_{\nu\lambda} - 2\partial_\gamma \Upsilon_{\mu\beta} \partial_\alpha \Upsilon_{\nu\lambda} + \partial_\mu \Upsilon_{\gamma\beta} \partial_\nu \Upsilon_{\alpha\lambda}] \quad (\text{B5})$$

or

$$\Gamma_{\mu\beta}^{\alpha(2)} \Gamma_{\nu\alpha}^{\beta(2)} = \frac{1}{4} \sinh^2(\Phi) [g^{\alpha\gamma} g^{\beta\lambda} \partial_\mu \Upsilon_{\gamma\beta} \partial_\nu \Upsilon_{\alpha\lambda} + 2(g^{\alpha\gamma} g^{\beta\lambda} - g^{\alpha\beta} g^{\gamma\lambda}) \partial_\beta \Upsilon_{\gamma\mu} \partial_\alpha \Upsilon_{\nu\lambda}]. \quad (\text{B6})$$

Finally we have the Ricci tensor given by:

$$\begin{aligned} R_{\mu\nu}^{(2)} &= \frac{1}{2} \sinh^2(\Phi) \partial_\alpha \Upsilon^{\alpha\beta} (\partial_\mu \Upsilon_{\beta\nu} + \partial_\nu \Upsilon_{\beta\mu} - \partial_\beta \Upsilon_{\mu\nu}) \\ &\quad + \frac{1}{2} \sinh(\Phi) g^{\alpha\beta} (\partial_\alpha \partial_\mu \Upsilon_{\beta\nu} + \partial_\alpha \partial_\nu \Upsilon_{\beta\mu} - \partial_\alpha \partial_\beta \Upsilon_{\mu\nu}) \\ &\quad - \frac{1}{4} \sinh^2(\Phi) [g^{\alpha\gamma} g^{\beta\lambda} \partial_\mu \Upsilon_{\gamma\beta} \partial_\nu \Upsilon_{\alpha\lambda} + 2(g^{\alpha\gamma} g^{\beta\lambda} - g^{\alpha\beta} g^{\gamma\lambda}) \partial_\beta \Upsilon_{\gamma\mu} \partial_\alpha \Upsilon_{\nu\lambda}]. \end{aligned} \quad (\text{B7})$$

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